Casimir effect for the scalar field under Robin boundary conditions: A functional integral approach

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(June 1, 2004)

Abstract

In this work we show how to define the action of a scalar field in a such a way that Robin boundary condition is implemented dynamically, i.e., as a consequence of the stationary action principle. We discuss the quantization of that system via functional integration. Using this formalism, we derive an expression for the Casimir energy of a massless scalar field under Robin boundary conditions on a pair of parallel plates, characterized by constants c_1 and c_2. Some special cases are discussed; in particular, we show that for some values of c_1 and c_2 the Casimir energy as a function of the distance between the plates presents a minimum. We also discuss the renormalization at one-loop order of the two-point Green function in the $\lambda\phi^4$ theory submitted to Robin boundary condition on a plate.

PACS numbers: 03.70.+k, 11.10.-z, 11.10.Kk, 11.10.Gh

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I. INTRODUCTION

The Casimir force between two uncharged macroscopic bodies in vacuum is widely regarded as arising from the zero-point fluctuations intrinsic to any quantum system. In the case of two flat parallel plates a distance a apart the pressure is proportional to a^{-4} [1]. In recent years several groups have performed high-precision measurements of the Casimir force between a flat plate and a spherical surface (lens) or a sphere [2], and also between two parallel flat plates [3]. In the last case the original Casimir formula was confirmed to 15% accuracy.

Due to its fundamental character the Casimir effect has applications in many areas of physics, ranging from the theory of elementary particles and interactions [4,5] to atomic and molecular physics [6]. Besides, it has analogues in condensed matter physics, for instance in fluctuation induced forces [7] and boundary critical phenomena [8]. More recently, its relevance to the design and operation of micro- and nano-scale electromechanical devices has been emphasized [9].

Usually the details of the interaction between the vacuum fluctuations of the quantum field and the macroscopic bodies are neglected, and replaced by classical boundary conditions (BC) at the boundary of the latter. While Dirichlet and Neumann BC have been extensively studied over the past, the more general case of Robin BC has attracted little attention.

A field ϕ is said to obey Robin boundary condition at a surface Σ if its normal derivative at a point on Σ is proportional to its value there:

$$\frac{\partial}{\partial n}\phi(x) = c\,\phi(x), \qquad x \in \Sigma. \tag{1}$$

Neumann and Dirichlet boundary conditions are particular cases of Robin boundary condition: the first one corresponds to $c = 0$; the other is obtained in the limit $c \to \infty$ (assuming that $\partial_n \phi$ is bounded).

The mixed case of Dirichlet-Robin (DR) BC were considered in [10] for a 2D massless scalar field as a phenomenological model for a penetrable surface, with c^{-1} playing the role of the finite penetration depth. Recently, the Casimir energy for a scalar field subject to Robin BC on one or two parallel planes was computed in [11].

Here we also compute the Casimir energy for a scalar field under Robin BC on two parallel planes. However, we introduce a rather different approach which seems more amenable to an eventual computation of radiative corrections in models containing interactions. Its starting point is the introduction of suitable boundary terms in the action, which allows us to compute the partition function of the system without the explicit imposition of Robin BC on the fields. In spite of that we show that the two-point Green function does satisfy those boundary conditions. We also find agreement with the main results of Ref. [11] for the Casimir energy, which was computed there using the more conventional approach of summing the zero-point energy of the normal modes of the field. In particular, we show that for the mixed case of DR boundary conditions the Casimir energy as a function of a develops a minimum, i.e., there is a configuration of stable equilibrium.

Finally, we study the renormalization at one-loop order of the two-point Green function in the $\lambda\phi^4$ theory submitted to Robin BC on a plate. Our analysis differs from previous ones [8,12] in two aspects: (i) we keep c arbitrary, instead of considering only the the particular

cases $c = 0$ or $c = \infty$, and (ii) we perform the regularization entirely in momentum space. This procedure avoids dealing with distributions and test functions, which are unavoidable in the mixed coordinate-momentum space regularization used in [8,12].

II. THE MODIFIED ACTION

Let us consider, for simplicity, a real scalar field living in the half-space $z \geq 0$,[1] satisfying the (Euclidean) equation of motion

$$-\partial^2 \phi + U'(\phi) = 0 \tag{2}$$

and subject to Robin boundary condition at $z = 0$,

$$\partial_z \phi - c\phi \Big|_{z=0} = 0. \tag{3}$$

(We shall assume that $c \geq 0$, in order to avoid the possible appearance of tachyons in the theory.) One can easily verify that Eqs. (2) and (3) are consequences of the stationary action principle applied to the Euclidean action

$$S[\phi] = \int d^d x \left\{ \int_0^\infty dz \left[\frac{1}{2} (\partial_\mu \phi)^2 + U(\phi) \right] + \frac{1}{2} c \phi^2(\mathbf{x}, 0) \right\}, \tag{4}$$

where

$$\int d^d x := \lim_{\beta \to \infty} \lim_{L \to \infty} \int_0^\beta dx^0 \int_{-L/2}^{L/2} dx^1 \cdots \int_{-L/2}^{L/2} dx^{d-1}. \tag{5}$$

Indeed, computing $\delta S := S[\phi + \eta] - S[\phi]$ up to second order in η we obtain

$$\delta S = \int d^d x \left\{ (-\partial_z \phi + c\phi) \eta \Big|_{z=0} + \int_0^\infty dz \left[-\partial^2 \phi + U'(\phi) \right] \eta \right\} + O(\eta^2), \tag{6}$$

which implies (2) and (3) if ϕ is a stationary point of S.

Until now, we have been talking about a *classical* field. What happens when one quantizes the theory? In the usual functional integrals approach one has to integrate over all field configurations obeying certain boundary conditions. If such is the case, is it necessary to retain the surface term in the action? Bordag *et al.* [13] argue that it is, in order to ensure the Hermiticity (more precisely, the self-adjointness) of the fluctuation operator $\hat{\mathcal{F}} := -\partial^2 + U''(\phi_c)$, where ϕ_c is the solution to Eqs. (2) and (3). Saharian [14] has also argued in favor of such a surface term: without it, the vacuum energy evaluated as the sum of the zero-point energy of each normal mode of the field does not agree with the result obtained by integrating the vacuum energy density.

Here we propose a different approach. We retain the surface term in the action, but we shall not impose any boundary condition at $z = 0$ on the field configurations to be

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[1] Conventions: $\hbar = c = 1$, $x = (\mathbf{x}, z)$, where $\mathbf{x} := (x^0, \ldots, x^{d-1})$ and $z := x^d$.

integrated over. Somewhat surprisingly, if we treat $U(\phi)$ as a perturbation, the two-point Green function of the unperturbed theory does satisfy Robin BC at $z = 0$, i.e.,

$$(\partial_z - c) \langle \phi(x) \, \phi(x') \rangle_0 \Big|_{z=0} = 0. \tag{7}$$

Let us prove it. First of all, we write the partition function of the unperturbed theory as[2]

$$Z_0 = \oint [D\phi_1] \int_{\phi(\mathbf{x},0)=\phi_1(\mathbf{x})} [D\phi] \, \exp(-S_0), \tag{8}$$

where S_0 is given by Eq. (4) without $U(\phi)$. Note that we are integrating over all field configurations satisfying the boundary condition $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$, and then we integrate over all configurations of the surface field $\phi_1(\mathbf{x})$. In other words, we integrate over all possible boundary conditions at $z = 0$. $\phi(\mathbf{x}, z)$ also satisfies periodic BC in the $\mathbf{x}$-coordinates, with period β in the x^0-direction and L in the others (note, however, that we have the limits $\beta \to \infty$ and $L \to \infty$ in mind).

We now decompose ϕ as a sum of two fields: $\phi = \phi_0 + \eta$, where ϕ_0 satisfies

$$\partial^2 \phi_0(x) = 0, \qquad \phi_0(\mathbf{x}, 0) = \phi_1(\mathbf{x}), \qquad \phi_0(\mathbf{x}, \infty) = 0. \tag{9}$$

Note that, because of the boundary conditions imposed on ϕ_0, η vanishes at $z = 0$ and $z = \infty$. Eq. (9) can be solved using Fourier transform; the result is

$$\phi_0(\mathbf{x}, z) = \int \frac{d^d k}{(2\pi)^d} \, e^{i\mathbf{k}\cdot\mathbf{x}} \, \varphi_1(\mathbf{k}) \, e^{-kz}, \tag{10}$$

where $k = |\mathbf{k}|$ and $\varphi_1(\mathbf{k})$ is the Fourier transform of $\phi_1(\mathbf{x})$.

In terms of the fields φ_1 and η, the partition function Z_0 becomes the product of two independent functional integrals: $Z_0 = Z_A Z_B$, where

$$Z_A = \int [D\varphi_1] \, \exp\left\{ -\int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \, (c + k) \, \varphi_1(\mathbf{k}) \, \varphi_1(-\mathbf{k}) \right\}, \tag{11}$$

$$Z_B = \int_{\eta(\mathbf{x},0)=0} [D\eta] \, \exp\left\{ -\int d^d x \int_0^\infty dz \, \frac{1}{2} \, (\partial_\mu \eta)^2 \right\}. \tag{12}$$

Let us now compute the two-point Green function. As a consequence of the factorization of Z_0, one has $\langle \phi(x) \, \phi(x') \rangle_0 = \langle \phi_0(x) \, \phi_0(x') \rangle_A + \langle \eta(x) \, \eta(x') \rangle_B$. According to Eq. (11) we have

$$\langle \varphi_1(\mathbf{k}) \, \varphi_1(\mathbf{k}') \rangle_A = \frac{1}{c + k} \, \delta^{(d)}(\mathbf{k} + \mathbf{k}'). \tag{13}$$

Combining this result with Eq. (10) we obtain

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[2]We follow here a procedure very similar to the one employed in Ref. [15] in the context of quantum field theory at finite temperature.

$$\langle \phi_0(x)\, \phi_0(x')\rangle_A = \int \frac{d^d k}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\, \frac{e^{-k(z+z')}}{c+k}\,. \tag{14}$$

On the other hand,

$$\langle \eta(x)\, \eta(x')\rangle_B = \int \frac{d^d k}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\, D_\eta(\mathbf{k}; z, z'), \tag{15}$$

where D_η satisfies

$$(-\partial_z^2 + k^2)\, D_\eta(\mathbf{k}; z, z') = \delta(z - z'), \qquad D_\eta(\mathbf{k}; 0, z') = D_\eta(\mathbf{k}; \infty, z') = 0. \tag{16}$$

One can easily verify that the solution to (16) is given by

$$D_\eta(\mathbf{k}; z, z') = \frac{1}{k}\, \sinh(kz_<)\, \exp(-kz_>), \tag{17}$$

where $z_<(z_>) = \min(\max)\{z, z'\}$. Collecting terms, we finally obtain

$$\langle \phi(x)\, \phi(x')\rangle_0 = \int \frac{d^d k}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \left[\frac{e^{-k(z+z')}}{c+k} + \frac{1}{k}\, \sinh(kz_<)\, \exp(-kz_>) \right]. \tag{18}$$

One can easily verify that $\langle \phi(x)\, \phi(x')\rangle_0$ indeed satisfies Robin BC at $z = 0$, Eq. (7).

III. CASIMIR ENERGY

Let us now apply our procedure to the computation of the Casimir energy of a free massless scalar field ϕ subject to Robin BC on two parallel plates located at the planes $z = 0$ and $z = a$,

$$\partial_z \phi - c_1 \phi \Big|_{z=0} = 0, \qquad \partial_z \phi + c_2 \phi \Big|_{z=a} = 0 \qquad (c_1, c_2 \geq 0). \tag{19}$$

The Euclidean action for such a system is given by

$$S = \int d^d x \left\{ \int_0^a dz \left[\frac{1}{2}\, (\partial_\mu \phi)^2 \right] + \frac{1}{2}\, c_1 \phi^2(\mathbf{x}, 0) + \frac{1}{2}\, c_2 \phi^2(\mathbf{x}, a) \right\}, \tag{20}$$

and its partition function is given by

$$Z = \oint [D\phi_1][D\phi_2] \int_{\phi(\mathbf{x},0)=\phi_1(\mathbf{x})}^{\phi(\mathbf{x},a)=\phi_2(\mathbf{x})} [D\phi]\, \exp(-S). \tag{21}$$

The Casimir energy can be extracted from Z using the identity

$$E_0 = -\lim_{\beta \to \infty} \frac{1}{\beta}\, \ln Z. \tag{22}$$

As we did in the previous Section, we shall write ϕ as the sum of two terms: $\phi = \phi_0 + \eta$, where ϕ_0 is the solution to the classical equation of motion, $\partial^2 \phi_0 = 0$, that obeys the

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boundary conditions $\phi_0(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ and $\phi_0(\mathbf{x}, a) = \phi_2(\mathbf{x})$ (consequently $\eta(\mathbf{x}, 0) = \eta(\mathbf{x}, a) = 0$). By Fourier transforming in the $\mathbf{x}$-coordinates one can explicitly solve for ϕ_0, obtaining

$$\phi_0(\mathbf{x}, z) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{\sinh ka} [\varphi_1(\mathbf{k}) \sinh k(a - z) + \varphi_2(\mathbf{k}) \sinh kz] \tag{23}$$

where $k = |\mathbf{k}|$ and $\varphi_j(\mathbf{k})$ is the Fourier transform of $\phi_j(\mathbf{x})$, $j = 1, 2$.

Expressing $S[\phi]$ in terms of ϕ_0 and η, we obtain $S = S_A + S_B$, where

$$S_A = \int d^d x \left\{ \int_0^a dz \left[\frac{1}{2} (\partial_\mu \phi_0)^2 \right] + \frac{1}{2} c_1 \phi_0^2(\mathbf{x}, 0) + \frac{1}{2} c_2 \phi_0^2(\mathbf{x}, a) \right\} \tag{24}$$

$$S_B = \int d^d x \int_0^a dz \left[\frac{1}{2} (\partial_\mu \eta)^2 \right] \tag{25}$$

Since ϕ_0 is a functional solely of ϕ_1 and ϕ_2, the partition function Z can be written as the product of two terms: $Z = Z_A Z_B$, where

$$Z_A = \oint [D\phi_1][D\phi_2] \exp(-S_A), \qquad Z_B = \int_{\eta(\mathbf{x}, 0)=0}^{\eta(\mathbf{x}, a)=0} [D\eta] \exp(-S_B). \tag{26}$$

It follows from Eq. (22) that the Casimir energy is given by the sum of two terms, $E_0 = E_A + E_B$, of which the second one is the Casimir energy of a field subject to *Dirichlet* boundary conditions on the plates. Since it is a well known result [16], we shall just quote the result:

$$E_B(a) = -\frac{L^{d-1}}{a^d} \Gamma\left(\frac{d+1}{2}\right) (4\pi)^{-(d+1)/2} \zeta(d+1). \tag{27}$$

Let us now compute E_A. Inserting (23) into (24) we can rewrite S_A in terms of φ_1 and φ_2 as

$$S_A = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \varphi_i(\mathbf{k}) M_{ij}(\mathbf{k}) \varphi_j(-\mathbf{k}), \tag{28}$$

$$M_{ij}(\mathbf{k}) = (c_i + k \coth ka) \delta_{ij} - k \operatorname{csch} ka (1 - \delta_{ij}). \tag{29}$$

Changing the variables of integration in Z_A to φ_j, we obtain

$$Z_A = \oint [D\varphi_1][D\varphi_2] \exp(-S_A[\varphi_1, \varphi_2]) = \prod_{\mathbf{k}} \det{}^{-1/2} M(\mathbf{k}), \tag{30}$$

hence

$$\begin{aligned} E_A^{\text{bare}}(a) &= \lim_{\beta \to \infty} \frac{1}{2\beta} \sum_{\mathbf{k}} \ln \det M(\mathbf{k}) \\ &= \frac{L^{d-1}}{2} \int \frac{d^d k}{(2\pi)^d} \ln\left[c_1 c_2 + k^2 + (c_1 + c_2) k \coth ka \right]. \end{aligned} \tag{31}$$

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The expression above diverges, hence requires renormalization. This is achieved by subtracting from it the quantity

$$\delta E_A = \lim_{a \to \infty} E_A^{\text{bare}}(a) = \frac{L^{d-1}}{2} \int \frac{d^d k}{(2\pi)^d} \ln[(c_1 + k)(c_2 + k)], \tag{32}$$

which can be interpreted as part of the self-energy of the plates. (An analogous subtraction is necessary in the calculation of E_B.) Since δE_A does not depend on the distance between the plates, it does not contribute to the force between them. Its subtraction from E_A^{bare} is thus permissible as long as one is interested — as we are — only in the Casimir force. The result of the subtraction is given by

$$E_A(a) = \frac{L^{d-1}}{2} \int \frac{d^d k}{(2\pi)^d} \ln\left[1 + \frac{2(c_1 + c_2)k}{(c_1 + k)(c_2 + k)} \cdot \frac{1}{e^{2ka} - 1}\right]. \tag{33}$$

Performing the angular integration and adding the result to E_B, Eq. (27), we finally obtain the Casimir energy for the massless scalar field under Robin boundary conditions:

$$E_0(c_1, c_2; a) = E_B(a) + \frac{L^{d-1}}{(4\pi)^{d/2}\, \Gamma\left(\frac{d}{2}\right)} \int_0^\infty dk\, k^{d-1} \ln\left[1 + \frac{2(c_1 + c_2)k}{(c_1 + k)(c_2 + k)} \cdot \frac{1}{e^{2ka} - 1}\right]. \tag{34}$$

As a check of this result, we note that the integral vanishes if $c_1 = c_2 \to \infty$ or $c_1 = c_2 = 0$, thus reproducing the correct result for Dirichlet-Dirichlet and Neumann-Neumann boundary conditions. Dirichlet-Neumann boundary conditions ($c_1 \to \infty$, $c_2 = 0$) can also be treated exactly: in this case, the integral in Eq. (34) becomes

$$I(a) := \int_0^\infty dk\, k^{d-1} \ln \coth ka. \tag{35}$$

Integrating by parts yields

$$I(a) = \frac{2a}{d} \int_0^\infty dk\, \frac{k^d}{\sinh 2ka} = \frac{4a}{d} \sum_{n=0}^\infty \int_0^\infty dk\, k^d\, e^{-2(2n+1)ka} = \left(2 - \frac{1}{2^d}\right) \frac{\Gamma(d)}{(2a)^d} \zeta(d+1). \tag{36}$$

Inserting this result into Eq. (34) and using the identity $\Gamma(2z) = (4\pi)^{-1/2}\, 2^{2z}\, \Gamma(z)\, \Gamma(z + 1/2)$ [18] we finally obtain

$$E_0(\infty, 0; a) = \frac{L^{d-1}}{a^d} \left(1 - \frac{1}{2^d}\right) \Gamma\left(\frac{d+1}{2}\right) (4\pi)^{-(d+1)/2} \zeta(d+1), \tag{37}$$

which agrees with the correct result [17], thus giving us another check of Eq. (34).

Next in simplicity are the following three cases: (i) $c_1 = c_2 = c$, (ii) $c_1 = \infty$, $c_2 = c$, and (iii) $c_1 = 0$, $c_2 = c$. We shall denote them RR, DR and NR, respectively (R = Robin, D = Dirichlet, and N = Neumann). In all these cases, changing the variable of integration in Eq. (34) to $q = k/c$ allows us to rewrite it as

$$E_0^\alpha(c, a) = L^{d-1} c^d\, \mathcal{E}_\alpha(ca) \qquad (\alpha = \text{RR}, \text{DR}, \text{NR}), \tag{38}$$

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where

$$\mathcal{E}_\alpha(x) = -\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{(4\pi)^{(d+1)/2}\,x^d} + \frac{1}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\int_0^\infty dq\,q^{d-1}\,\ln\left[1 + \frac{f_\alpha(q)}{e^{2qx}-1}\right], \qquad (39)$$

with

$$f_{\mathrm{RR}}(q) = \frac{4q}{(1+q)^2}\,, \qquad f_{\mathrm{DR}}(q) = \frac{2q}{1+q}\,, \qquad f_{\mathrm{NR}}(q) = \frac{2}{1+q}\,. \qquad (40)$$

The graphs of $\mathcal{E}_\alpha(ca)$ in three spatial dimensions are depicted in Fig. 1. We can conclude from it that the Casimir force between the plates: (i) is purely attractive in the RR case (i.e., $c_1 = c_2 = c$); (ii) is repulsive at short distances and attractive at long distances in the DR case, and (iii) is attractive at short distances and repulsive at long distances in the NR case.

To understand those behaviors, let us consider a free field subject to Robin BC at $z = 0$, i.e., $\partial_z\phi(\mathbf{x}, 0) = c\phi(\mathbf{x}, 0)$. If we write the z-dependent part of ϕ as $\varphi(z) = \sin(kz + \delta)$, the previous equation becomes $\tan\delta = k/c$. It follows that $\delta \to 0$ as $k \to 0$, and $\delta \to \pi/2$ as $k \to \infty$. In terms of $\varphi(z)$, this is equivalent to say that Robin BC tends to Dirichlet BC at low momentum, and to Neumann BC at high momentum. In the jargon of renormalization group theory, $c = \infty$ is an infrared and $c = 0$ is an ultraviolet attractive fixed point.

Let us now return to Fig. 1. According to the analysis above, the RR curve should behave as the DD curve in the infrared (i.e., $a \to \infty$) and as the NN curve in the ultraviolet (i.e., $a \to 0$). In both cases, the Casimir force is purely attractive, and so it is in the RR case. The DR curve should behave as the DD curve as $a \to \infty$, and as the DN curve as $a \to 0$; indeed, this is what we observe: attraction at long distances and repulsion at short distances. The analysis of the NR curve is similar.

Such considerations suggest an interesting possibility. Let us suppose that $0 < c_1 \ll c_2 < \infty$. Then $E_0(c_1, c_2; a) \sim E_0(\infty, \infty; a)$ as $a \to \infty$, and $E_0(c_1, c_2; a) \sim E_0(0, 0; a)$ as $a \to 0$. In both these limits, therefore, the Casimir force is expected to be attractive. However, since the crossover from a Dirichlet-like to a Neumann-like BC takes place at different scales for each plate, there could be a range of distances for which one has a Dirichlet-like BC at one plate and a Neumann-like BC at the other, thus leading to a repulsive force between them. That such a possibility can indeed occur is shown in Fig. 2.

IV. INTERACTING FIELD

There is still another reason for the introduction of boundary terms in the action: the perturbative treatment of a renormalizable interacting theory requires boundary counterterms in addition to the usual ones [8,12]. Thus, from a conceptual point of view, it is more natural to treat boundary conditions as resulting from the interaction of the field with a background than to impose them *a priori*. We shall illustrate this point with the calculation of the first order correction to the two-point Green function of the $\lambda\phi^4$ theory in the presence of a flat boundary, where the field is submitted to Robin BC.

The renormalized Euclidean Lagrangian density of the theory is given by[3]

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + c \, \delta(z) \, \phi^2 + \frac{\lambda}{4!} \, \phi^4 + \mathcal{L}_{\mathrm{ct}}, \tag{41}$$

with the field ϕ living in the half-space $z \geq 0$. For simplicity, we shall assume $m = 0$ and $c \geq 0$. $\mathcal{L}_{\mathrm{ct}}$ contains the renormalization counterterms — the usual ones,

$$\mathcal{L}_{\mathrm{ct,\,bulk}} = \frac{\delta Z}{2} \left(\partial_\mu \phi \right)^2 + \frac{\delta m^2}{2} \, \phi^2 + \frac{\delta \lambda}{4!} \, \phi^4, \tag{42}$$

plus boundary conterterms, which we shall exhibit later.

As we have seen in Sec. II, the unperturbed two-point Green function (Feynman propagator) is given by [see Eq. (18)]

$$G_0(x, x') := \langle \phi(x) \, \phi(x') \rangle_0 = \int \frac{d^d k}{(2\pi)^d} \, e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \, \mathcal{G}_0(k; z, z'), \tag{43}$$

where, in the massless theory,

$$\mathcal{G}_0(k; z, z') = \frac{e^{-k(z+z')}}{c + k} + \frac{1}{k} \, \sinh(kz_<) \, \exp(-kz_>). \tag{44}$$

If we neglect boundary counterterms, the first order correction to $\mathcal{G}_0$ is given by

$$\mathcal{G}_1(k; z, z') = - \int_0^\infty dw \, \mathcal{G}_0(k; z, w) \, \Sigma_1(w) \, \mathcal{G}_0(k; w, z'), \tag{45}$$

where the one-loop self-energy, including the mass counterterm, is given by

$$\begin{aligned}
\Sigma_1(w) &= \frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \, \mathcal{G}_0(q; w, w) + \delta m^2 \\
&= \frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \left(\frac{e^{-2qw}}{c + q} + \frac{1 - e^{-2qw}}{2q} \right) + \delta m^2.
\end{aligned} \tag{46}$$

(An ultraviolet cutoff Λ is implicit in the integral above and wherever necessary below.) To fix δm^2 we impose that $\Sigma_1(w)$ be finite for $w > 0$ and vanishes for $w \to \infty$ (the latter condition is equivalent to requiring that the physical mass is zero when one is infinitely far from the plate). These conditions imply

$$\delta m^2 = -\frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \, \frac{1}{2q}, \tag{47}$$

hence

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[3]The factor of $\frac{1}{2}$ in the boundary term in the action (4) is due to the fact that $\int_0^\infty \delta(z) \, f(z) \, dz = \frac{1}{2} f(0)$.

$$\Sigma_1(w) = \frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{c+q} - \frac{1}{2q} \right) e^{-2qw}. \tag{48}$$

Although $\Sigma_1(w)$ is now a well defined function of w for $w > 0$, this is not enough to ensure the finiteness of $\mathcal{G}_1$. Indeed, $\Sigma_1(w) \sim w^{-(d-1)}$ for $w \to 0$, causing the divergence of the integral in Eq. (45) for $d \geq 2$. This problem can be (partially) solved by the boundary counterterm

$$\mathcal{L}_{\text{ct, boundary}}^{(1)} = \delta c\, \delta(z)\, \phi^2. \tag{49}$$

It adds a new term to the self-energy $\Sigma_1(w)$, turning it into $\tilde{\Sigma}_1(w) = \Sigma_1(w) + 2\delta c\, \delta(w)$. Equation (45) then gets replaced by

$$\mathcal{G}_1(k; z, z') = -\int_0^\infty dw\, \mathcal{G}_0(k; z, w)\, \Sigma_1(w)\, \mathcal{G}_0(k; w, z') - \delta c\, \mathcal{G}_0(k; z, 0)\, \mathcal{G}_0(k; 0, z'). \tag{50}$$

Let us assume that $0 < z < z'$; then Eq. (50) yields

$$\mathcal{G}_1(k; z, z') = -\delta c\, \frac{e^{-k(z+z')}}{(c+k)^2} - \frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{c+q} - \frac{1}{2q} \right) [I(0, z) + I(z, z') + I(z', \infty)], \tag{51}$$

where

$$I(a, b) := \int_a^b dw\, e^{-2qw}\, \mathcal{G}_0(k; z, w)\, \mathcal{G}_0(k; w, z'). \tag{52}$$

Since we are interested in the large-q behaviour of the integrand, we may restrict our attention to $I(0, z)$ — $I(z, z')$ and $I(z', \infty)$ are exponentially supressed in that limit. Performing the integration, we obtain

$$\begin{aligned} I(0, z) &= e^{-k(z+z')} \int_0^z dw\, e^{-2qw} \left[\frac{e^{-kw}}{c+k} + \frac{1}{k} \sinh(kw) \right]^2 \\ &\stackrel{q\to\infty}{\sim} \frac{e^{-k(z+z')}}{(c+k)^2} \left[\frac{1}{2q} + \frac{c}{2q^2} + O\left(\frac{1}{q^3} \right) \right]. \end{aligned} \tag{53}$$

If we further multiply this result by the term in parentheses in Eq. (51), we conclude that the integrand in that equation behaves for large q as

$$\frac{e^{-k(z+z')}}{(c+k)^2} \left[\frac{1}{4q^2} - \frac{c}{4q^3} + O\left(\frac{1}{q^4} \right) \right] \stackrel{q\to\infty}{\sim} \frac{e^{-k(z+z')}}{(c+k)^2} \left[\frac{1}{4q(q+c)} + O\left(\frac{1}{q^4} \right) \right]. \tag{54}$$

Therefore, if we choose

$$\delta c = -\frac{\lambda}{8} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q(q+c)} + \delta\bar{c}, \tag{55}$$

with $\delta\bar{c}$ finite, $\mathcal{G}_1(k; z, z')$ becomes finite for $d < 4$.

To see why the boundary conterterm (49) only solves the problem partially, let us take $z = z' = 0$ in Eq. (50). A straightforward calculation then shows that

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$$\mathcal{G}_1(k;0,0) = -\frac{1}{(c+k)^2} \left[\delta c + \frac{\lambda}{2} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{c+q} - \frac{1}{2q} \right) \frac{1}{2(q+k)} \right]$$

$$= -\frac{\delta \overline{c}}{(c+k)^2} + \frac{\lambda}{8(c+k)} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q(q+c)(q+k)}. \tag{56}$$

The result is finite for $d = 2$, but diverges for $d = 3$. In this case, another boundary counterterm is needed. As we show below, it is given by

$$\mathcal{L}_{\text{ct, boundary}}^{(2)} = \delta b \, \delta(z) \, \phi(\partial_n - c)\phi, \tag{57}$$

with ∂_n the interior normal derivative (in our case, $\partial_n = \partial_z$). Indeed, such a counterterm gives rise to an extra contribution to $\mathcal{G}_1(k;0,0)$, given by

$$\Delta \mathcal{G}_1(k;0,0) = -\delta b \int_0^\infty dw \, \delta(w) \, (\partial_w - 2c) \, \mathcal{G}_0^2(k;0,w)$$

$$= -\delta b \int_0^\infty dw \, \delta(w) \, (\partial_w - 2c) \, \frac{e^{-2kw}}{(c+k)^2}$$

$$= \frac{\delta b}{c+k}. \tag{58}$$

If we choose

$$\delta b = -\frac{\lambda}{8} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q+c)} + \delta \overline{b}, \tag{59}$$

with $\delta \overline{b}$ finite, we cancel the divergence in $\mathcal{G}_1(k;0,0)$ for $d = 3$. Indeed,

$$\mathcal{G}_1(k;0,0) + \Delta \mathcal{G}_1(k;0,0) = \frac{1}{c+k} \left[\delta \overline{b} - \frac{\delta \overline{c}}{c+k} - \frac{\lambda k}{8} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q+c)(q+k)} \right]$$

$$= \frac{1}{c+k} \left[\delta \overline{b} - \frac{\delta \overline{c}}{c+k} + \frac{\lambda}{16\pi^2} \frac{k}{c-k} \ln\left(\frac{k}{c}\right) \right] \qquad (d=3). \tag{60}$$

The boundary counterterm (57) is ineffective if $z, z' > 0$; in this case, $\Delta \mathcal{G}_1(k;z,z')$ is identically zero. (As a consequence, $\mathcal{G}_1(k;z,z')$ is insentive to the choice of $\delta \overline{b}$ if $z, z' > 0$.) On the other hand, that counterterm is necessary if $z = 0$ and $z' > 0$ (or vice-versa); in this case, it is possible to show that the same choice (59) for δb also ensures the finiteness of $\mathcal{G}_1(k;z,z')$.

In order to fix the finite part of the boundary counterterms, $\delta \overline{b}$ and $\delta \overline{c}$, one has to impose a pair of renormalization conditions. A natural choice, for it is satisfied at tree level, is given by

$$\mathcal{G}(\kappa;0,0) = (c+\kappa)^{-1}, \tag{61}$$

$$\frac{d}{dk} \mathcal{G}(k;0,0) \bigg|_{k=\kappa} = -(c+\kappa)^{-2}, \tag{62}$$

where κ is some arbitrary, but nonzero, mass scale.

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We end this section with some remarks:

(A) Because of the boundary counterterms, the renormalized two-point Green function *does not* satisfy Robin BC at $z = 0$ — except for $c \to \infty$ (Dirichlet BC); in this case, the boundary condition is preserved at each order in perturbation theory [12].

(B) Inclusion of another plate at $z = a$ does not affect significantly the overall picture. New ultraviolet divergences arise, but the theory is made finite with the same type of counterterms used in the case of a single plate.

V. CONCLUSIONS

In this work we have computed the Casimir energy of a free massless scalar field subject to independent Robin boundary conditions on two parallel plates in d spatial dimensions. It was shown that for mixed Dirichlet-Robin BC the Casimir energy as a function of the distance a between the plates displays a minimum. We managed to understand the behavior of the Casimir energy as a function of c relying on an analogy with the renormalization group flows expected from infrared/ultraviolet fixed points. It was found that the Dirichlet BC ($c \to \infty$) is analogous to an attractive infrared fixed point whereas the Neumann BC ($c = 0$) resembles an attractive ultraviolet fixed point. This interpretation is consistent with the numerical results shown in Fig. 1 for the Casimir energy as a function of ca, and suggests that a crossover from Dirichlet-like to Neumann-like behaviors at the plates may lead to a repulsive force between them.

We also provided a detailed analysis of renormalization for the two-point Green function $G(x, x')$ at first order in λ in the $\lambda\phi^4$ theory. For simplicity we worked out the case of Robin BC at a single flat boundary. We have shown that, in addition to the usual "bulk" counterterms, one boundary counterterm is necessary to render G finite in the bulk, and a second one is necessary if at least one of its arguments lies on the boundary. This analysis is a necessary step in the computation of radiative corrections to the Casimir energy, which we intend to present elsewhere.

In view of the widespread interest on the Casimir effect and its possible technological applications it is worthwhile to seek alternative computational tools which may go beyond or complement the existing ones. For instance, in [19] a resummation scheme to compute the leading radiative corrections to the Casimir energy was suggested. On the other hand, the method outlined in the present work seems well suited to the computation of radiative corrections either in a perturbative setting or eventually via application of semi-classical methods. Its starting point is the indirect implementation of the BC by means of appropriate terms in the action functional, which is then reexpressed in terms of two kinds of fields: a field $\eta(\mathbf{x}, z)$ satisfying Dirichlet BC at the surfaces, and two surface fields $\phi_j(\mathbf{x})$ localized on the planes, depending only on the remaining d transverse coordinates. The advantage of this procedure is that the functional integration over the surface fields is unconstrained, i.e., one does not have to enforce explicitly the Robin BC on the fields.

The implementation of BC via local terms in the action is usually employed in studies of boundary critical phenomena [8]. In that context, it can be shown that Dirichlet and Neumann BC correspond to the so-called ordinary ($c \to \infty$) and special transitions ($c = 0$), respectively. The Robin BC is relevant in the study of the crossover between those

universality classes, for which, however, the computations become much more involved. It is relevant also for the analysis of the ordinary transition; in this case, however, one may resort to an expansion in powers of c^{-1} [8]. We expect that the methods proposed here can be useful to the study of the crossover for the relevant case of two flat planes. This is presently under investigation.

ACKNOWLEDGMENTS

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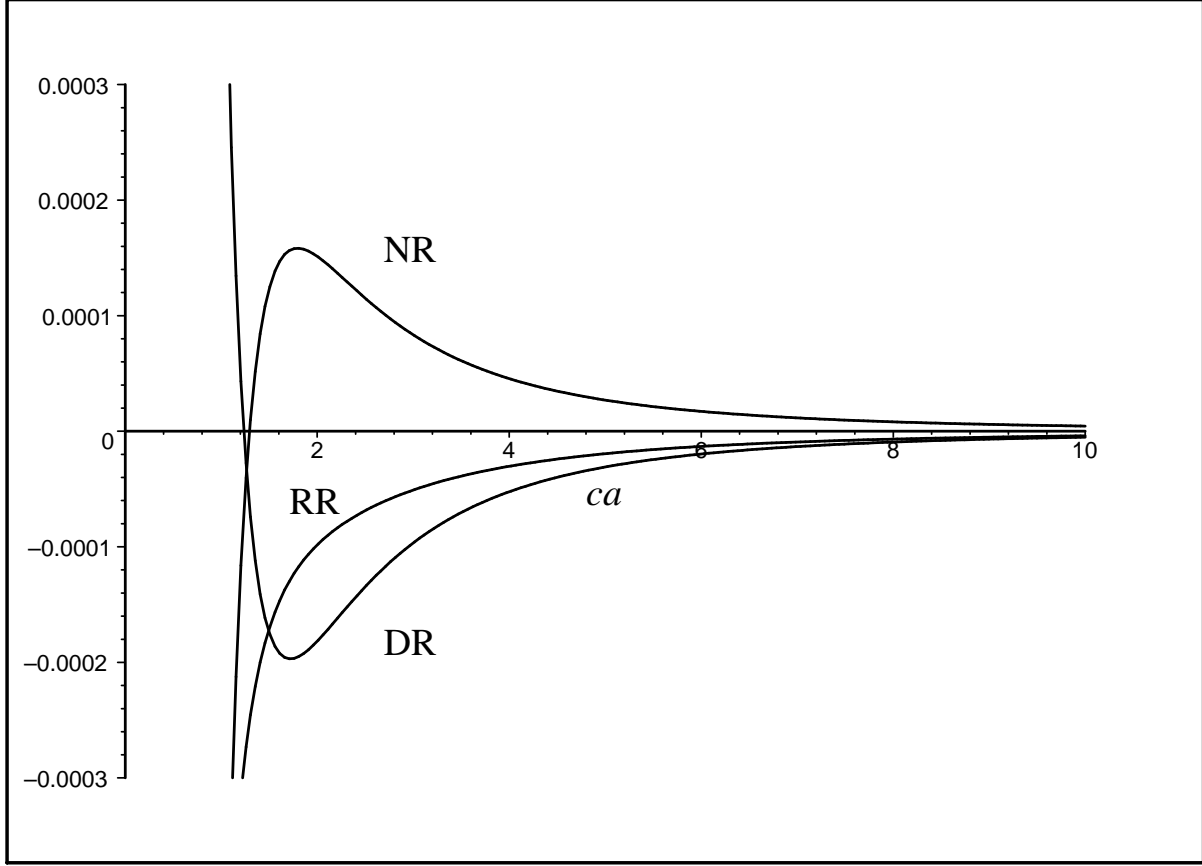[19] de Albuquerque L C and Cavalcanti R M 2002 *Phys. Rev.* D **65** 045004

FIG. 1. $\mathcal{E}_\alpha(ca)$ [see Eq. (38)] in three spatial dimensions for $\alpha = $ RR ($c_1 = c_2 = c$), $\alpha = $ DR ($c_1 = \infty$, $c_2 = c$), and $\alpha = $ NR ($c_1 = 0$, $c_2 = c$).
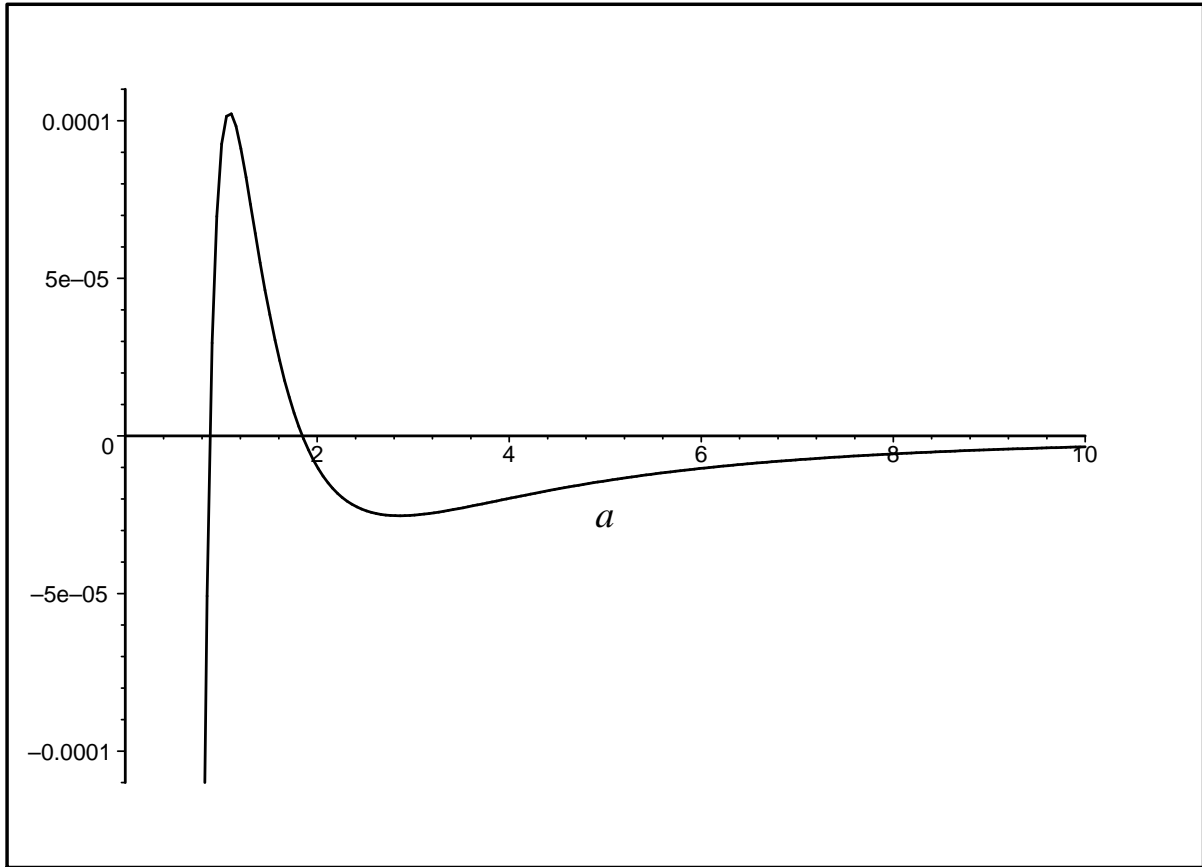
FIG. 2. Casimir energy per unit area *vs*. distance between plates ($c_1 = 1/2$, $c_2 = 2$, $d = 3$).